

# Cauchy-Sylvester's theorem on compound determinants and modules of differential operators on Coxeter arrangements

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## Abstract

We prove that the modules of differential operators of order 2 on the classical Coxeter arrangements are free by exhibiting bases. For this purpose, we use Cauchy-Sylvester's theorem on compound determinants and Saito-Holm's criterion. In the case type  $A$ , we apply Cauchy-Sylvester's theorem on compound determinants to Vandermonde determinant. By using the Schur polynomials, we define operators which form a part of a basis of modules of differential operators on the classical Coxeter arrangements of type  $A$ . In the cases of type  $B$  and type  $D$ , the proofs go similarly to the case of type  $A$  with some adjustments of operators and determinants.

**Key Words:** Coxeter arrangement; Cauchy-Sylvester's compound determinants; Schur functions.

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## 1 Introduction

Let  $K$  be a field of characteristic zero, and let  $V$  be an  $\ell$ -dimensional vector space over  $K$ . Let  $\{x_1, \dots, x_\ell\}$  be a basis for the dual space  $V^*$ , and let  $S := \text{Sym}(V^*) \simeq K[x_1, \dots, x_\ell]$  be the polynomial ring. Let  $D^{(m)}(S) := \bigoplus_{|\alpha|=m} S \partial^\alpha$  be the module of differential operators (of order  $m$ ) of  $S$ , where  $\alpha \in \mathbb{N}^\ell$  is a multi-index. A nonzero element  $\theta = \sum_{|\alpha|=m} f_\alpha \partial^\alpha \in D^{(m)}(S)$  is homogeneous of degree  $i$  if  $f_\alpha$  is zero or homogeneous of degree  $i$  for each  $\alpha$ .

When  $\theta \in D^{(m)}(S)$  is homogeneous of degree  $i$ , we write  $\deg(\theta) = i$ . For a multi-index  $\alpha$ , we put

$$x_\alpha := (x_1, \dots, x_1, x_2, \dots, x_2, \dots, x_\ell, \dots, x_\ell), \quad (1.1)$$

where the number of  $x_i$  is  $\alpha_i$ .

Let  $\mathcal{A}$  be a central (hyperplane) arrangement (i.e., every hyperplane contains the origin) in  $V$ . For each hyperplane  $H \in \mathcal{A}$  fix a linear form  $p_H \in V^*$  such that  $\ker(p_H) = H$ , and put  $Q(\mathcal{A}) := \prod_{H \in \mathcal{A}} p_H$ . We call  $Q(\mathcal{A})$  a defining polynomial of  $\mathcal{A}$ . We define the **module**  $D^{(m)}(\mathcal{A})$  of  $\mathcal{A}$ -**differential operators** of order  $m$  as follows:

$$D^{(m)}(\mathcal{A}) := \{ \theta \in D^{(m)}(S) \mid \theta(Q(\mathcal{A})S) \subseteq Q(\mathcal{A})S \}.$$

In the case  $m = 1$ ,  $D^{(1)}(\mathcal{A})$  is the module of  $\mathcal{A}$ -derivations. We say  $\mathcal{A}$  to be free if  $D^{(1)}(\mathcal{A})$  is a free  $S$ -module. An excellent reference on arrangements is the book by Orlik and Terao [5].

The classical Coxeter arrangements  $\mathcal{A}_{\ell-1}$ ,  $\mathcal{B}_\ell$  and  $\mathcal{D}_\ell$  of type  $A$ ,  $B$  and  $D$  are defined as

$$\begin{aligned} \mathcal{A}_{\ell-1} &:= \{ H_{ij} = \{x_i - x_j = 0\} \mid 1 \leq i < j \leq \ell \}, \\ \mathcal{B}_\ell &:= \{ H_i = \{x_i = 0\} \mid i = 1, \dots, \ell \} \cup \{ H_{ij}^{\pm 1} = \{x_i \pm x_j = 0\} \mid 1 \leq i < j \leq \ell \}, \\ \mathcal{D}_\ell &:= \{ H_{ij}^{\pm 1} = \{x_i \pm x_j = 0\} \mid 1 \leq i < j \leq \ell \}. \end{aligned}$$

It is well-known that the Coxeter arrangements are free (see Theorem 6.60 in [5]). There are some other interesting reaserch for Coxeter arrangements. Orlik-Terao [6], for example, proved that a restriction of the Coxeter arrangement to each lattice element is free. One of other interesting reaserch for Coxeter arrangement is the study of freeness of Coxeter multiarrangements. Coxeter multiarrangements were studied by Solomon-Terao [8], Terao [9] and so on. In contrast, no one has yet been able to put the study of the module of differential operators on the Coxeter arrangement into practice.

There exists a well-known basis for  $D^{(1)}(\mathcal{A})$  when  $\mathcal{A}$  is one of the classical Coxeter arrangements (see for example [4]). The aim of this paper is to prove that the modules of differential operators of order 2 on the classical Coxeter arrangements are free by constructing bases. For this purpose, we introduce Cauchy-Sylvester's theorem on compound determinants and Saito-Holm's criterion.

In Section 3, we give some applications of the Cauchy-Sylvester's theorem on compound determinants. By using the results of Section 3, we will show that the modules of differential operators of order 2 on the classical Coxeter arrangements are free in Section 4 and 5.

## 2 Saito-Holm's criterion

In this section, we explain the Saito-Holm criterion. Put  $s_m := \binom{\ell+m-1}{m}$  and  $t_m := \binom{\ell+m-2}{m-1}$ , and set

$$\{\alpha^{(1)}, \dots, \alpha^{(s_m)}\} = \{\alpha \in \mathbb{N}^\ell \mid |\alpha| = m\},$$

where  $|\alpha| = \alpha_1 + \dots + \alpha_\ell$  for a multi-index  $\alpha \in \mathbb{N}^\ell$ . For operators  $\theta_1, \dots, \theta_{s_m} \in D^{(m)}(\mathcal{A})$ , define the **coefficient matrix**  $M_m(\theta_1, \dots, \theta_{s_m})$  of the operators  $\theta_1, \dots, \theta_{s_m}$  as follows:

$$M_m(\theta_1, \dots, \theta_{s_m}) := \left( \theta_i \left( \frac{x^{\alpha^{(j)}}}{\alpha^{(j)}!} \right) \right)_{1 \leq i, j \leq s_m},$$

where  $\alpha! = \alpha_1! \cdots \alpha_\ell!$ . Thus the  $(i, j)$ -entry of the coefficient matrix is the polynomial coefficient of  $\partial^{\alpha^{(j)}}$  in  $\theta_i$ .

The following criterion was originally given by Saito [7] in the case  $m = 1$ , and was generalized by Holm [1] into the case  $m$  general.

**Proposition 2.1** (Saito-Holm's criterion). *Let  $\theta_1, \dots, \theta_{s_m} \in D^{(m)}(\mathcal{A})$  be homogeneous operators. Then the following two conditions are equivalent:*

- (1)  $\det M_m(\theta_1, \dots, \theta_{s_m}) = cQ^{t_m}$  for some  $c \in K^\times$ .
- (2)  $\theta_1, \dots, \theta_{s_m}$  form a basis for  $D^{(m)}(\mathcal{A})$  over  $S$ .

When  $D^{(m)}(\mathcal{A})$  is a free  $S$ -module, we define the **exponents** of  $D^{(m)}(\mathcal{A})$  to be the multiset of degrees of a homogeneous basis  $\{\theta_1, \dots, \theta_{s_m}\}$  for  $D^{(m)}(\mathcal{A})$ , which is denoted by  $\exp D^{(m)}(\mathcal{A})$ :

$$\exp D^{(m)}(\mathcal{A}) = \{\deg(\theta_1), \dots, \deg(\theta_{s_m})\}.$$

### 3 Cauchy-Sylvester's theorem on compound determinants

Throughout this paper, assume  $\ell \geq m$ . In this section, we will follow the notation of the paper by Ito and Okada [3] as far as possible. We denote by  $\succ$  the lexicographic order on  $\mathbb{Z}^m$ . That is, for  $\mu = (\mu_1, \dots, \mu_m)$  and  $\nu = (\nu_1, \dots, \nu_m) \in \mathbb{Z}^m$ , we write  $\mu \succ \nu$  if there exist an index  $k$  such that

$$\mu_1 = \nu_1, \dots, \mu_{k-1} = \nu_{k-1}, \text{ and } \mu_k > \nu_k.$$

Put

$$Z := \{\mu = (\mu_1, \dots, \mu_m) \in \mathbb{Z}^m \mid 1 \leq \mu_1 < \mu_2 < \dots < \mu_m \leq \ell\}.$$

Then  $Z$  is a totally ordered subset of  $\mathbb{Z}^m$ . Put  $x_\mu := (x_{\mu_1}, \dots, x_{\mu_m}) \in S^m$ .

Let  $A = (a_{i,j})_{1 \leq i,j \leq \ell}$  be a square matrix of order  $\ell$ . For  $\mu, \nu \in Z$  put

$$A_{\mu,\nu} := (a_{\mu_i,\nu_j})_{1 \leq i,j \leq m}.$$

We define the  $m$ -th **compound matrix**  $A^{(m)}$  by

$$A^{(m)} := (\det A_{\mu,\nu})_{\mu,\nu \in Z},$$

where the rows and columns are arranged in the increasing order on  $Z$ .

The following was obtained by Cauchy and Sylvester (see for example [3, Proposition 3.1]).

**Proposition 3.1** (Cauchy-Sylvester). *Let  $A = (a_{i,j})_{1 \leq i,j \leq \ell}$  be a square matrix. Then the determinant of the  $m$ -th compound matrix  $A^{(m)}$  is given by*

$$\det A^{(m)} = (\det A)^{\binom{\ell-1}{m-1}}. \quad (3.1)$$

Put

$$\Lambda := \{\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{Z}^m \mid \ell - m \geq \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m \geq 0\}.$$

We regard  $\Lambda$  as a totally ordered subset of  $\mathbb{Z}^m$  by the order  $\succ$ . Then the map

$$Z \ni (\mu_1, \dots, \mu_m) \mapsto (\ell - m + 1 - \mu_1, \ell - m + 2 - \mu_2, \dots, \ell - \mu_m) \in \Lambda$$

is a bijection between  $\Lambda$  and  $Z$ , and this bijection reverses the ordering on  $\Lambda$  and  $Z$ .

For  $\lambda \in \Lambda$ , we define the following symmetric polynomials and a Laurent polynomial:

$$s_\lambda^A := \frac{\det(t_i^{\lambda_j+m-j})_{1 \leq i,j \leq m}}{\det(t_i^{m-j})_{1 \leq i,j \leq m}} \in S[t_1, \dots, t_m], \quad (3.2)$$

$$s_\lambda^B := \frac{\det(t_i^{2(\lambda_j+m-j)+1})_{1 \leq i,j \leq m}}{\det(t_i^{2(m-j)})_{1 \leq i,j \leq m}} \in S[t_1, \dots, t_m], \quad (3.3)$$

$$s_\lambda^D := \frac{\det(t_i^{2(\lambda_j+m-j)-1})_{1 \leq i,j \leq m}}{\det(t_i^{2(m-j)})_{1 \leq i,j \leq m}} \in S[t_1^{\pm 1}, \dots, t_m^{\pm 1}]. \quad (3.4)$$

The polynomial  $s_\lambda^A$  is the Schur polynomial corresponding to the partition  $\lambda$ . We remark that  $s_\lambda^D$  is a symmetric polynomial if  $\lambda_m \geq 1$ . Now the degrees of these Laurent polynomials are as follows:

$$\deg s_\lambda^A = |\lambda|, \quad \deg s_\lambda^B = 2|\lambda| + m, \quad \deg s_\lambda^D = 2|\lambda| - m, \quad (3.5)$$

where  $|\lambda| := \lambda_1 + \dots + \lambda_m$ .

**Proposition 3.2.** *We have the following determinant identities:*

$$\det(s_\lambda^A(x_\mu))_{\substack{\lambda \in \Lambda \\ \mu \in Z}} = \left[ \prod_{1 \leq i < j \leq \ell} (x_i - x_j) \right]^{\binom{\ell-2}{m-1}}, \quad (3.6)$$

$$\det(s_\lambda^B(x_\mu))_{\substack{\lambda \in \Lambda \\ \mu \in Z}} = (x_1 \cdots x_\ell)^{\binom{\ell-1}{m-1}} \left[ \prod_{1 \leq i < j \leq \ell} (x_i^2 - x_j^2) \right]^{\binom{\ell-2}{m-1}}, \quad (3.7)$$

$$\det(s_\lambda^D(x_\mu))_{\substack{\lambda \in \Lambda \\ \mu \in Z}} = \frac{1}{(x_1 \cdots x_\ell)^{\binom{\ell-1}{m-1}}} \left[ \prod_{1 \leq i < j \leq \ell} (x_i^2 - x_j^2) \right]^{\binom{\ell-2}{m-1}}. \quad (3.8)$$

*Proof.* Apply the formula (3.1) to the matrices  $A = (x_i^{\ell-j})_{1 \leq i,j \leq \ell}$ ,  $A = (x_i^{2(\ell-j)+1})_{1 \leq i,j \leq \ell}$  and  $A = (x_i^{2(\ell-j)-1})_{1 \leq i,j \leq \ell}$ .  $\square$

We will use these determinant identities to prove that  $D^{(2)}(\mathcal{A})$  is free when  $\mathcal{A}$  is a classical Coxeter arrangement in Section 4 and 5.

## 4 Type $A$ and $B$

Let  $\mathcal{A}$  be an arbitrary arrangement. By [2, Proposition 2.3] and [2, Theorem 2.4], we have

$$D^{(m)}(\mathcal{A}) = \bigcap_{H \in \mathcal{A}} D^{(m)}(p_H S), \quad (4.1)$$

where  $D^{(m)}(p_H S) = \{\theta \in D^{(m)}(S) \mid \theta(p_H x^\alpha) \in p_H S \text{ for any } |\alpha| = m-1\}$  for  $H \in \mathcal{A}$ .

Recall that the defining polynomials of Coxeter arrangements  $\mathcal{A}_{\ell-1}$  and  $\mathcal{B}_\ell$  of types  $A$  and  $B$  are

$$\begin{aligned} Q(\mathcal{A}_{\ell-1}) &= \prod_{1 \leq i < j \leq \ell} (x_i - x_j), \\ Q(\mathcal{B}_\ell) &= x_1 \cdots x_\ell \prod_{1 \leq i < j \leq \ell} (x_i^2 - x_j^2). \end{aligned}$$

We introduce some operators in  $D^{(m)}(\mathcal{A}_{\ell-1})$  and  $D^{(m)}(\mathcal{B}_\ell)$ . By using these operators, we construct bases for the modules  $D^{(2)}(\mathcal{A}_{\ell-1})$  and  $D^{(2)}(\mathcal{B}_\ell)$  of differential operators of order 2 on  $\mathcal{A}_{\ell-1}$  and  $\mathcal{B}_\ell$ .

Let  $k = 1, \dots, \ell$ , and put  $h_k^A := (x_k - x_1) \cdots (x_k - x_{k-1})(x_k - x_{k+1}) \cdots (x_k - x_\ell)$  and  $h_k^B := x_k(x_k^2 - x_1^2) \cdots (x_k^2 - x_{k-1}^2)(x_k^2 - x_{k+1}^2) \cdots (x_k^2 - x_\ell^2)$ . We define operators  $\eta_k^A$  and  $\eta_k^B$  in  $D^{(m)}(S)$  as follows:

$$\eta_k^A := h_k^A \frac{1}{m!} \partial_k^m, \quad \eta_k^B := h_k^B \frac{1}{m!} \partial_k^m.$$

Then  $\deg \eta_k^A = \ell - 1$  and  $\deg \eta_k^B = 2\ell - 1$ .

It is convenient to write  $f \doteq g$  for  $f, g \in S$  if  $f = cg$  for some  $c \in K^\times$ .

**Proposition 4.1.** *For  $k = 1, \dots, \ell$ , we have that  $\eta_k^A \in D^{(m)}(\mathcal{A}_{\ell-1})$  and  $\eta_k^B \in D^{(m)}(\mathcal{B}_\ell)$ .*

*Proof.* For any  $1 \leq i < j \leq \ell$  and a multi-index  $\beta$  with  $|\beta| = m-1$ ,

$$\frac{1}{m!} \partial_k^m ((x_i \pm x_j) x^\beta) = \begin{cases} 1 & \text{if } i = k \text{ and } \beta_i + 1 = m, \\ \pm 1 & \text{if } j = k \text{ and } \beta_j + 1 = m, \\ 0 & \text{otherwise.} \end{cases}$$

If  $i = k$  and  $\beta_i + 1 = m$  or  $j = k$  and  $\beta_j + 1 = m$ , then  $\eta_k^A((x_i - x_j)x^\beta) \doteq h_k^A \in (x_i - x_j)S$ . Therefore we obtain  $\eta_k^A \in D^{(m)}(\mathcal{A}_{\ell-1})$  from the formula (4.1).

Similarly we have  $\eta_k^B \in \bigcap_{1 \leq i < j \leq \ell} D^{(m)}((x_i^2 - x_j^2)S)$ . For  $i = 1, \dots, \ell$  and a multi-index  $\beta$  with  $|\beta| = m - 1$ , we have

$$\eta_k^B((x_i)x^\beta) = \begin{cases} h_k^B & \text{if } i = k \text{ and } \beta_i + 1 = m, \\ 0 & \text{otherwise.} \end{cases}$$

This leads to that  $\eta_k^B \in \bigcap_{i=1}^\ell D^{(m)}(x_i S)$ . Therefore we obtain  $\eta_k^B \in D^{(m)}(\mathcal{B}_\ell)$ .  $\square$

For a Laurent polynomial  $f(t_1, \dots, t_m) \in S[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$  satisfying  $f(x_\alpha) \in S$  for any  $\alpha$  with  $|\alpha| = m$ , we define an operator

$$\theta_f := \sum_{|\alpha|=m} f(x_\alpha) \frac{1}{\alpha!} \partial^\alpha.$$

We call a Laurent polynomial  $f(t_1, \dots, t_m)$  is symmetric if

$$f(t_1, \dots, t_i, \dots, t_j, \dots, t_m) = f(t_1, \dots, t_j, \dots, t_i, \dots, t_m)$$

for all pairs  $(i, j)$ .

**Lemma 4.2.** *Assume that  $f(t_1, \dots, t_m)$  is a symmetric Laurent polynomial. Then we have that  $\theta_f \in D^{(m)}(\mathcal{A}_{\ell-1})$ .*

*Proof.* Since  $f(t_1, \dots, t_m)$  is symmetric, we have

$$\theta_f((x_i - x_j)x^\beta) \big|_{x_i=x_j} = (f(x_{\beta+e_i}) - f(x_{\beta+e_j})) \big|_{x_i=x_j} = 0$$

for any  $1 \leq i < j \leq \ell$  and a multi-index  $\beta$  with  $|\beta| = m - 1$ . We obtain  $\theta_f((x_i - x_j)x^\beta) \in (x_i - x_j)S$ . Hence it follows from (4.1) that  $\theta_f \in D^{(m)}(\mathcal{A}_{\ell-1})$ .  $\square$

For  $\lambda \in \Lambda$ , define operators

$$\theta_\lambda^A := \sum_{|\alpha|=m} s_\lambda^A(x_\alpha) \frac{1}{\alpha!} \partial^\alpha, \quad \theta_\lambda^B := \sum_{|\alpha|=m} s_\lambda^B(x_\alpha) \frac{1}{\alpha!} \partial^\alpha.$$

Then  $\deg \theta_\lambda^A = |\lambda|$ ,  $\deg \theta_\lambda^B = 2|\lambda| + m$  by the formula (3.5).

**Proposition 4.3.** For  $\lambda \in \Lambda$ , we have  $\theta_\lambda^A \in D^{(m)}(\mathcal{A}_{\ell-1})$  and  $\theta_\lambda^B \in D^{(m)}(\mathcal{B}_\ell)$ .

*Proof.* Since Laurent polynomials  $s_\lambda^A$  and  $s_\lambda^B$  are symmetric, we obtain  $\theta_\lambda^A, \theta_\lambda^B \in D^{(m)}(\mathcal{A}_{\ell-1})$  by Lemma 4.2.

By the formula (4.1), we can write

$$D^{(m)}(\mathcal{B}_\ell) = D^{(m)}(\mathcal{A}_{\ell-1}) \cap \left( \bigcap_{i=1}^{\ell} D^{(m)}(x_i S) \right) \cap \left( \bigcap_{1 \leq i < j \leq \ell} D^{(m)}((x_i + x_j)S) \right).$$

Thus we only need to prove that

$$\theta_\lambda^B \in \left( \bigcap_{i=1}^{\ell} D^{(m)}(x_i S) \right) \quad \text{and} \quad \theta_\lambda^B \in \left( \bigcap_{1 \leq i < j \leq \ell} D^{(m)}((x_i + x_j)S) \right).$$

For any  $i = 1, \dots, \ell$  and a multi-index  $\beta$  with  $|\beta| = m - 1$ , we have

$$\theta_\lambda^B(x_i x^\beta) = s_\lambda^B(x_{\beta+e_i}) = x_i x^\beta s_\lambda^A(x_{\beta+e_i}^2) \in x_i S.$$

This implies  $\bigcap_{i=1}^{\ell} \theta_\lambda^B \in D^{(m)}(x_i S)$ .

For any  $1 \leq i < j \leq \ell$  and a multi-index  $\beta$  with  $|\beta| = m - 1$ ,

$$\theta_\lambda^B((x_i + x_j)x^\beta) = s_\lambda^B(x_{\beta+e_i}) + s_\lambda^B(x_{\beta+e_j}) = x^\beta \left( x_i s_\lambda^A(x_{\beta+e_i}^2) + x_j s_\lambda^A(x_{\beta+e_j}^2) \right)$$

Then we have  $\theta_\lambda^B((x_i + x_j)x^\beta)|_{x_i=-x_j} = 0$ , and this implies  $\theta_\lambda^B((x_i + x_j)x^\beta) \in (x_i + x_j)S$ . Hence we obtain  $\theta_\lambda^B \in D^{(m)}(\mathcal{B}_\ell)$ .  $\square$

**Theorem 4.4.** Let  $m = 2$ .

(1) The set

$$C_A := \{\eta_i^A \mid i = 1, \dots, \ell\} \cup \{\theta_\lambda^A \mid \lambda \in \Lambda\}$$

forms an  $S$ -basis for  $D^{(2)}(\mathcal{A}_{\ell-1})$ . Hence

$$\exp D^{(2)}(\mathcal{A}_{\ell-1}) = \{\ell - 1, \dots, \ell - 1\} \cup \{|\lambda| \mid \lambda \in \Lambda\}.$$

(2) The set

$$C_B := \{\eta_i^B \mid i = 1, \dots, \ell\} \cup \{\theta_\lambda^B \mid \lambda \in \Lambda\}$$

forms an  $S$ -basis for  $D^{(2)}(\mathcal{B}_\ell)$ . Hence

$$\exp D^{(2)}(\mathcal{B}_\ell) = \{2\ell - 1, \dots, 2\ell - 1\} \cup \{2|\lambda| + 2 \mid \lambda \in \Lambda\}.$$



*Proof.* (1) All operators in  $C_{\mathcal{A}}$  belong to  $D^{(2)}(\mathcal{A}_{\ell-1})$  by Proposition 4.1 and Proposition 4.3.

By Proposition 2.1, we only need to prove that the determinant of the coefficient matrix  $M_m(C_{\mathcal{A}})$  of the operators of  $C_{\mathcal{A}}$  is equal to  $Q(\mathcal{A}_{\ell-1})^\ell$  up to a nonzero constant. By Proposition 3.2, we obtain  $\det (s_\lambda^{\mathcal{A}}(x_\alpha))_{\lambda \in \Lambda, \alpha \in Z} = Q(\mathcal{A})^{\ell-2}$ . Hence we have

$$\det M_m(C_{\mathcal{A}}) \doteq Q(\mathcal{A}_{\ell-1})^2 \begin{vmatrix} I_\ell & \\ 0 & \det (s_\lambda^{\mathcal{A}}(x_\alpha))_{\substack{\lambda \in \Lambda \\ \alpha \in Z}}^* \end{vmatrix} = Q(\mathcal{A}_{\ell-1})^2 \cdot Q(\mathcal{A}_{\ell-1})^{\ell-2} = Q(\mathcal{A}_{\ell-1})^\ell.$$

(2) We have an identity

$$\det M_m(C_{\mathcal{B}}) \doteq x_1 \cdots x_\ell \left( \prod_{1 \leq i < j \leq \ell} (x_i^2 - x_j^2) \right)^2 \begin{vmatrix} I_\ell & \\ 0 & \det (s_\lambda^{\mathcal{B}}(x_\alpha))_{\substack{\lambda \in \Lambda \\ \alpha \in Z}}^* \end{vmatrix} = Q(\mathcal{B}_\ell)^\ell$$

by Proposition 3.2. Then the rest of proof for (2) is similar to the one for (1).  $\square$

## 5 Type $D$

In this section, we assume  $m = 2$ , and we construct a basis for  $D^{(2)}(\mathcal{D}_\ell)$ . Recall the defining polynomial  $Q(\mathcal{D}_\ell) = \prod_{1 \leq i < j \leq \ell} (x_i^2 - x_j^2)$  of the Coxeter arrangement of type  $D$ .

Set

$$\begin{aligned} \Lambda' &:= \{ \lambda = (\lambda_1, \lambda_2) \mid \ell - 2 \geq \lambda_1 \geq \lambda_2 \geq 1 \}, \\ \Lambda'' &:= \{ \lambda = (\lambda_1, \lambda_2) \mid \ell - 2 \geq \lambda_1 \geq 0, \lambda_2 = 0 \}. \end{aligned}$$

Then  $\Lambda = \Lambda' \cup \Lambda''$ . Put  $\lambda^{(0)} := (0, 0)$ . We define operators  $\theta_\lambda^{\mathcal{D}}$  as follows:

$$\begin{aligned} \theta_\lambda^{\mathcal{D}} &:= \sum_{|\alpha|=2} s_\lambda^{\mathcal{D}}(x_\alpha) \frac{1}{\alpha!} \partial^\alpha \quad \text{if } \lambda \in \Lambda', \\ \theta_\lambda^{\mathcal{D}} &:= (x_1 \cdots x_\ell) \sum_{|\alpha|=2} s_\lambda^{\mathcal{D}}(x_\alpha) \frac{1}{\alpha!} \partial^\alpha \quad \text{if } \lambda \in \Lambda'' \setminus \{\lambda^{(0)}\}, \\ \theta_\lambda^{\mathcal{D}} &:= (x_1 \cdots x_\ell)^2 \sum_{|\alpha|=2} s_\lambda^{\mathcal{D}}(x_\alpha) \frac{1}{\alpha!} \partial^\alpha \quad \text{if } \lambda = \lambda^{(0)}. \end{aligned}$$

If  $\lambda \in \Lambda'$ , then we have

$$s_\lambda^{\mathcal{D}} = \frac{\det(t_i^{2(\lambda_j-1+2-j)+1})_{1 \leq i, j \leq 2}}{\det(t_i^{2(2-j)})_{1 \leq i, j \leq 2}} = s_{\lambda-\mathbf{1}}^{\mathcal{B}},$$

where  $\lambda - \mathbf{1} = (\lambda_1 - 1, \lambda_2 - 1)$ .

If  $\lambda \in \Lambda'' \setminus \{\lambda^{(0)}\}$ , then

$$s_\lambda^{\mathcal{D}} = \frac{t_1^{2\lambda_1+1} \cdot t_2^{-1} - t_2^{2\lambda_1+1} \cdot t_1^{-1}}{t_1^2 - t_2^2} = \frac{1}{t_1 t_2} \sum_{j=0}^{\lambda_1} t_1^{2j} t_2^{2(\lambda_1-j)}.$$

Thus  $(x_1 \cdots x_\ell) s_\lambda^{\mathcal{D}}(x_\alpha)$  is a polynomial for any multi-index  $\alpha$  with  $|\alpha| = 2$ .

We have

$$\theta_{\lambda^{(0)}}^{\mathcal{D}} = (x_1 \cdots x_\ell)^2 \left( \sum_{i=1}^{\ell} \frac{1}{2x_i^2} \partial_i^2 + \sum_{1 \leq i < j \leq \ell} \frac{1}{x_i x_j} \partial_i \partial_j \right).$$

Hence  $\theta_\lambda^{\mathcal{D}}$  for any  $\lambda \in \Lambda$ . The degrees of these operators are as follows:

$$\begin{aligned} \deg \theta_\lambda^{\mathcal{D}} &= 2|\lambda| - 2 = 2\lambda_1 + 2\lambda_2 - 2 \quad \text{if } \lambda \in \Lambda', \\ \deg \theta_\lambda^{\mathcal{D}} &= 2\lambda_1 - 2 + \ell \quad \text{if } \lambda \in \Lambda'' \setminus \{\lambda^{(0)}\}, \\ \deg \theta_\lambda^{\mathcal{D}} &= 2\ell - 2 \quad \text{if } \lambda = \lambda^{(0)}. \end{aligned}$$

**Proposition 5.1.** *For  $\lambda \in \Lambda$ , we have  $\theta_\lambda^{\mathcal{D}} \in D^{(2)}(\mathcal{D}_\ell)$ .*

*Proof.* By Lemma 4.2, we have  $\theta_\lambda^{\mathcal{D}} \in D^{(2)}(\mathcal{A}_{\ell-1})$  for any  $\lambda \in \Lambda$ .

Since

$$\begin{aligned} & \left( \sum_{|\alpha|=2} s_\lambda^{\mathcal{D}}(x_\alpha) \frac{1}{\alpha!} \partial^\alpha \right) ((x_i + x_j)x_k) \\ &= s_\lambda^{\mathcal{D}}(x_i, x_k) + s_\lambda^{\mathcal{D}}(x_j, x_k) \\ &= \frac{1}{x_i x_k} s_\lambda^{\mathcal{A}}(x_i^2, x_k^2) + \frac{1}{x_j x_k} s_\lambda^{\mathcal{A}}(x_j^2, x_k^2) \\ &= \frac{1}{x_i x_j x_k} (x_j s_\lambda^{\mathcal{A}}(x_i^2, x_k^2) + x_i s_\lambda^{\mathcal{A}}(x_j^2, x_k^2)), \end{aligned}$$

we obtain  $\theta_\lambda^{\mathcal{D}}((x_i + x_j)x_k)|_{x_i=-x_j} = 0$  for  $1 \leq i < j \leq \ell$ ,  $k = 1, \dots, \ell$  and  $\lambda \in \Lambda$ . Hence we have  $\theta_\lambda^{\mathcal{D}} \in D^{(2)}(\mathcal{D})$  for any  $\lambda \in \Lambda$ .  $\square$

We introduce other operators  $h_k^{\mathcal{D}}$  of  $D^{(2)}(\mathcal{D}_\ell)$ . For  $k = 1, \dots, \ell$  put  $h_k^{\mathcal{D}} := (x_k^2 - x_1^2) \cdots (x_k^2 - x_{k-1}^2)(x_k^2 - x_{k+1}^2) \cdots (x_k^2 - x_\ell^2)$ , and define

$$\eta_k^{\mathcal{D}} := \frac{h_k^{\mathcal{D}}}{2x_k} \partial_k^2 - (-1)^{\ell-1} \frac{1}{x_k} \theta_{\lambda^{(0)}}^{\mathcal{D}}.$$

The coefficient of  $\partial_k^2$  in  $\eta_k^{\mathcal{D}}$  is

$$\frac{h_k^{\mathcal{D}}}{2x_k} - (-1)^{\ell-1} \frac{(x_1 \cdots x_\ell)^2}{2x_k \cdot x_k^2} = \frac{h_k^{\mathcal{D}} - (-1)^{\ell-1} (x_1 \cdots x_{k-1} x_{k+1} \cdots x_\ell)^2}{2x_k} \in S.$$

Hence we obtain  $\eta_k^{\mathcal{D}} \in D^{(2)}(S)$ , and  $\deg \eta_k^{\mathcal{D}} = 2\ell - 2$ .

**Proposition 5.2.** *For  $k = 1, \dots, \ell$ , we have that  $\eta_k^{\mathcal{D}} \in D^{(2)}(\mathcal{D}_\ell)$ .*

*Proof.* Let  $k = 1, \dots, \ell$ . It is clear that  $\frac{h_k^{\mathcal{D}}}{2} \partial_k^2 \in D^{(2)}(\mathcal{D}_\ell)$ , and we have  $\theta_{\lambda^{(0)}}^{\mathcal{D}} \in D^{(2)}(\mathcal{D}_\ell)$  by Proposition 5.1. Thus we have  $\frac{h_k^{\mathcal{D}}}{2} \partial_k^2 - (-1)^{\ell-1} \theta_{\lambda^{(0)}}^{\mathcal{D}} \in D^{(2)}(\mathcal{D}_\ell)$ . This leads to  $\eta_k^{\mathcal{D}} \in D^{(2)}(\mathcal{D}_\ell)$ .  $\square$

**Theorem 5.3.** *Assume  $m = 2$ . The set*

$$C_{\mathcal{D}} := \{\eta_i^{\mathcal{D}} \mid i = 1, \dots, \ell\} \cup \{\theta_\lambda^{\mathcal{D}} \mid \lambda \in \Lambda\}$$

*forms an  $S$ -basis for  $D^{(2)}(\mathcal{D}_\ell)$ . Hence*

$$\begin{aligned} \exp D^{(2)}(\mathcal{D}_\ell) = & \{2\ell - 2, \dots, 2\ell - 2\} \cup \{2\lambda_1 + 2\lambda_2 - 2 \mid \ell - 2 \geq \lambda_1 \geq \lambda_2 \geq 1\} \\ & \cup \{2\lambda_1 - 2 + \ell \mid \ell - 2 \geq \lambda_1 \geq 1\} \cup \{2\ell - 2\}. \end{aligned}$$

*Proof.* By Proposition 5.1 and Proposition 5.2, we have  $C_{\mathcal{D}} \subseteq D^{(2)}(\mathcal{D}_\ell)$ . Let  $M_2(C_{\mathcal{D}})$  be the coefficient matrix of the operators in  $C_{\mathcal{D}}$ . We shall show that  $\det M_2(C_{\mathcal{D}}) \doteq Q(\mathcal{D}_\ell)^\ell$ .

Put  $\theta_\lambda := \sum_{|\alpha|=2} s_\lambda^{\mathcal{D}}(x_\alpha) \frac{1}{\alpha!} \partial^\alpha$  for  $\lambda \in \Lambda$ . Then

$$\begin{aligned} \det M_2(C_{\mathcal{D}}) &= \det M_2(\eta_i^{\mathcal{D}}, \theta_\lambda^{\mathcal{D}} \mid i = 1, \dots, \ell, \lambda \in \Lambda) \\ &= \det M_2(\eta_i^{\mathcal{D}} + (-1)^{\ell-1} \frac{1}{x_i} \theta_{\lambda^{(0)}}^{\mathcal{D}}, \theta_\lambda^{\mathcal{D}} \mid i = 1, \dots, \ell, \lambda \in \Lambda) \\ &= (x_1 \cdots x_\ell)^\ell \det M_2(\eta_i^{\mathcal{D}} + (-1)^{\ell-1} \frac{1}{x_i} \theta_{\lambda^{(0)}}^{\mathcal{D}}, \theta_\lambda \mid i = 1, \dots, \ell, \lambda \in \Lambda) \\ &\doteq \left( \frac{h_1^{\mathcal{D}}}{x_1} \cdots \frac{h_\ell^{\mathcal{D}}}{x_\ell} \right) (x_1 \cdots x_\ell)^\ell \left| \begin{array}{c|c} I_\ell & * \\ \hline 0 & \det (s_\lambda^{\mathcal{D}}(x_\alpha))_{\substack{\lambda \in \Lambda \\ \alpha \in Z}} \end{array} \right| \\ &= Q(\mathcal{D}_\ell)^2 (x_1 \cdots x_\ell)^{\ell-1} \frac{Q(\mathcal{D}_\ell)^{\ell-2}}{(x_1 \cdots x_\ell)^{\ell-1}} = Q(\mathcal{D}_\ell)^\ell \end{aligned}$$

by Proposition 3.2. Hence we conclude that the set  $C_{\mathcal{D}}$  forms an  $S$ -basis for  $D^{(2)}(\mathcal{D}_{\ell})$  by Proposition 2.1.  $\square$

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